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On the cascade in fully developed turbulence. The propagator approach versus the Markovian description

P.-O. Amblard[1,a] and J.-M. Brossier[1]

Laboratoire des Images et des Signaux, Groupe non linéaire [b], LIS-ENSIEG, BP 46, 38402 Saint-Martin d'Hères Cedex, France

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Abstract. The aim of the paper is to a provide a link between two approaches that describe the cascade in fully developed turbulence. Precisely, we show that the propagator approach (B. Castaing) and the Markovian description (Friedrich & Peinke) are equivalent in the log-normal case. To prove this, we use an Itô stochastic differential equation in scale – Friedrich & Peinke approach – that can be explicitly integrated. The solution corresponds to the propagator approach. This is done for scale-invariant cascades as well as non scale-invariant cascade.

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1 Introduction

Recent works on fully developed turbulence have focused on the explanation of the energy cascade from large to small scales and on the problem of intermittency. Two *a priori* different models have been developed: the propagator approach and the Markovian description. Although these approaches seem different, the aim of the paper is to establish their equivalence. We now recall the two approaches, beginning with the propagator approach.

In a series of papers [1–3], Castaing developed the propagator approach to describe the behavior of the probability density function (p.d.f.) of the velocity increments in turbulent flows. At large scales r, velocity increments $\delta v_r(t) = v(t+r) - v(t)$ measured on one component of the velocity roughly follow a Gaussian distribution. But when scale r is diminished, a curious phenomenon occurs: the p.d.f. of the velocity increments exhibits heavier and heavier tails – this fact contradicts Kolmogorov's K41 theory [4]. This could be an explanation of intermittency. To take into account this experimental result, Castaing proposed to relate the p.d.f. on one scale to a mixing of the p.d.f. on a larger scale. Precisely, he assumed that there exists a sequence $\eta < r_k < r_{k-1} < \ldots < r_1 < r_0 = L$ of scales such that

$$p_n(x) = \int G_{n,n-1}(\alpha) p_{n-1}\left(\frac{x}{\alpha}\right) \frac{\mathrm{d}\alpha}{\alpha} \qquad (1)$$

where p_n stands for the p.d.f. of the velocity increments at scale r_n, and where $G_{n,n-1}$ may be called the propagator. If $G_{n,n-1}$ does not depend on scales, then it is an infinitely divisible p.d.f. This hypothesis has been checked with some success on real experiments [5–7]. Note that equation (1) is a so-called affine convolution, and the resulting p.d.f. is the p.d.f. of the product of two independent random variables. This leads to the term multiplicative cascade. In this interpretation, the velocity increments are random variables that cascade from large to small scales *via* multiplication by a random variable. We will call indifferently *propagator* the random variable or its p.d.f.

Another approach has been developed more recently by Friedrich and Peinke [8–10]. Their idea was to describe the properties of the cascade by a Fokker-Planck equation. They verified on a set of experimental data that the so-called Kramers-Moyal coefficients [11] of order greater or equal to three vanish when estimated on the velocity increments. They also verified that the p.d.f. of the velocity increments satisfies correctly a Chapman-Kolmogorov equation through scales (in the inertial range). The conclusion of their works is that the cascade in turbulence is correctly described by a Markovian process, characterized by a drift and a diffusion term. Thus, the p.d.f. of the velocity increments satisfies a Fokker-Planck equation in scale.

In [12] p. 223, Castaing shows that in the log-normal case, a similar Fokker-Planck equation can be derived from the propagator approach. Furthermore, Donkov *et al.* recently proved that the solution of the Fokker-Planck equation proposed by Friedrich and Peinke satisfies an integral equation which is nothing but the propagator description of Castaing [13].

In this paper, we establish the same kind of links, taking as a starting point a stochastic differential equation in scale. Note that this approach has also been followed

[a] e-mail: Bidou.Amblard@lis.inpg.fr

[b] UPRESA CNRS 5083

recently by Marcq and Naert to describe the energy cascade [14].

This paper is organized as follows. In the next section, we recall some basic facts on stochastic differential equations and introduce our notations. Then, we describe the properties of the cascade as a stochastic differential equation and establish the link between the Castaing and the Friedrich and Peinke approaches. Then, we give the extension of the formalism to non scale-invariant cascades, and we finish the paper by discussing some points developed here.

2 Itô stochastic differential equations

Differential equations that involve white Gaussian noise are usually referred to as Langevin equations. However, integrating such equations leads to difficult problems – mostly because Brownian motion is almost surely not differentiable –, and a formalism due to Itô puts stochastic differential equations into a well-defined mathematical framework [15]. Let x_t be a stochastic process whose evolution is governed by

$$\mathrm{d}x_t = f(x_t,t)\mathrm{d}t + g(x_t,t)\mathrm{d}w_t \tag{2}$$

where w_t is a Brownian motion such that $E[w_1^2]=1$ ($E[.]$ stands for mathematical expectation or set average). $\mathrm{d}w_t$ represents the infinitesimal increments of the Brownian motion. When f and g are "good" functions, it can be shown that given a random initial condition x_{t_0}, x_t is a well-defined process. The previous equation is an Itô equation and must be undersdood in the sense of:

$$x_t = x_{t_0} + \int_{t_0}^{t} f(x_u,u)\mathrm{d}u + \int_{t_0}^{t} g(x_u,u)\mathrm{d}w_u$$

where $\int g(x_u,u)\mathrm{d}w_u$ is a stochastic Itô integral [15].

Note that other interpretations of stochastic differential equations exist. Among these, the most famous interpretation is that of Stratonovitch that leads to a different solution of the stochastic differential equation [15,16]. However, Itô and Stratonovitch differential equations are linked via Itô lemma, and going from Itô interpretation to Stratonovitch interpretation amounts to modify the drift coefficient. Furthermore, when working with drift and diffusion coefficients of a Fokker-Planck equation, the most natural interpretation is that of Itô [16].

Equation (2) shows that x_t is a Markov process. Furthermore, since w_t is Gaussian, the probability density function (p.d.f.) p_x of x_t, given the initial condition, satisfies Fokker-Planck equation

$$\partial_t p_x(u,t) = -\partial_u\left[f(u,t)p_x(u,t)\right] + \frac{1}{2}\partial^2_{uu}\left[g(u,t)^2 p_x(u,t)\right]$$

where ∂_y stands for $\partial/\partial y$.

We also recall that for a square integrable function h

$$E\left[\left(\int h(u)\mathrm{d}w_u\right)^2\right] = \int h(u)^2\mathrm{d}u.$$

We will be working with quantities defined in scales. We could work with velocity increments $\delta v_r(t) = v(t+r) - v(t)$ at scale r, but we prefer to work with the wavelet transform. For real valued signals, it is defined as

$$\mathcal{D}_a(t) = \int x_u \frac{1}{a}\psi\left(\frac{u-t}{a}\right)\mathrm{d}u$$

where a is the scale, and ψ is the wavelet – remember that to be a wavelet, ψ must have a vanishing integral –. The physical interpretation of the wavelet transform is that of a mathematical microscope: the lower the analysing scale, the finer the analysed structures. For the use of the wavelet transform in turbulence, we refer to [17] and references therein.

Setting aside the time variable (we will come back to this point in the discussion), we now study a Markov process in scale defined on the wavelet coefficients $\mathcal{D}_a$.

3 A Markovian process in scale

The works of Friedrich and Peinke show that the cascade in fully developed turbulence may be well-described by a diffusion Markov process. In other words, the cascade is well-described by an Itô stochastic differential equation. Furthermore, the experimental results described in [8,9] show that the drift and diffusion coefficients may be considered as affine functions of the velocity increments.

Therefore, we consider the following Itô stochastic differential equation that rules the dynamics of the wavelet coefficients through scales.

$$\mathrm{d}\mathcal{D}_a = (\alpha(a)\mathcal{D}_a + \beta(a))\mathrm{d}a + (\gamma(a)\mathcal{D}_a + \delta(a))\mathrm{d}w_a. \tag{3}$$

This equation starts with the initial condition $\mathcal{D}_{a_0}$ which is a random variable. Note that scale goes from a large scale (a_0) to small scales. a_0 is usually called the integral scale at which energy is injected. Experiments show that wavelet coefficients on large scales are nearly Gaussian, and therefore, $\mathcal{D}_{a_0}$ can be chosen to be Gaussian. Furthermore, a evolves in the inertial range $[\eta, a_0]$, where η is the so-called Kolmogorov scale where dissipation due to viscosity becomes dominant.

Note that equation (3) is a nonlinear stochastic differential equation (as expected when dealing with turbulence!). Although nonlinear, the equation is rather simple: it is a so-called bilinear stochastic differential equation, thus termed because of the presence of $\mathcal{D}_a\mathrm{d}w_a$. In this description, the cascade is viewed as the diffusion of the wavelet coefficients through scales.

Of importance with (3) is that, although nonlinear, it can be integrated [16]. The explicit solution of equation (3) is given by

$$\mathcal{D}_a = \Phi_{a,a_0}\left(\mathcal{D}_{a_0} + \int_{a_0}^{a}\Phi^{-1}_{s,a_0}\left[\beta(s)-\gamma(s)\delta(s)\right]\mathrm{d}s \right. \tag{4}$$

$$\left. + \int_{a_0}^{a}\Phi^{-1}_{s,a_0}\delta(s)\mathrm{d}w_s\right) \tag{5}$$

where

$$\Phi_{a,a_0} = \exp\left\{ \int_{a_0}^{a} \left(\alpha(s) - \frac{1}{2}\gamma(s)^2 \right) \mathrm{d}s + \int_{a_0}^{a} \gamma(s) \mathrm{d}w_s \right\}$$

and where integrals of the type $\int f(s)\mathrm{d}w_s$ are Itô stochastic integrals.

We now study the particular case $\beta(a) = 0$ and $\delta(a) = 0$. The solution of the equation is now simple, since it reads

$$\mathcal{D}_a = \Phi_{a,a_0} \mathcal{D}_{a_0} \quad (6)$$

$$\Phi_{a,a_0} = \exp\left\{ \int_{a_0}^{a} \left(\alpha(s) - \frac{1}{2}\gamma(s)^2 \right) \mathrm{d}s + \int_{a_0}^{a} \gamma(s) \mathrm{d}w_s \right\}$$

Φ_{a,a_0} is clearly a log-normal random variable. Furthermore, considering that

1) Castaing's result [12] p. 233 holds: in the log-normal case, the propagator implies a Fokker-Planck equation with linear drift and diffusion coefficients,
2) equation (6) holds

we conclude to the equivalence in the log-normal case of Friedrich and Peinke's and Castaing's approches of the cascade in turbulence.

Let $\Phi_{a,a_0} = \exp(W_{a,a_0})$. Then W_{a,a_0} is a Gaussian random variable whose mean and variance are

$$m(a, a_0) = \int_{a_0}^{a} \left(\alpha(s) - \frac{1}{2}\gamma(s)^2 \right) \mathrm{d}s \quad (7)$$

$$\sigma^2(a, a_0) = \int_{a_0}^{a} \gamma(s)^2 \mathrm{d}s. \quad (8)$$

Furthermore, we know that the probability density function $p_{\mathcal{D}}(u, a)$ of the wavelet coefficients on scale a satisfies the Fokker-Planck equation

$$-\partial_a p_{\mathcal{D}}(u, a) = -\partial_u \left[\alpha(a) u p_{\mathcal{D}}(u, a) \right] + \frac{1}{2}\partial_{uu}^2 \left[\gamma^2(a) u^2 p_{\mathcal{D}}(u, a) \right]$$

and that moments of $\mathcal{D}_a$, provided they exist, satisfy

$$-\partial_a E[\mathcal{D}_a^q] = qE[\mathcal{D}_a^q] \left\{ \alpha(a) + \frac{(q-1)}{2}\gamma^2(a) \right\}. \quad (9)$$

This equation is easily found from the Fokker-Planck equation. Note the "minus" sign in front of the scale partial derivative: we are going from large scales to small scales.

Suppose now that we look for a multifractal solution of (3). Then we know that the higher-order moments of the wavelet coefficients follow a scaling law of the form $a^{\zeta(q)}$, see for example [17]. Inserting $a^{\zeta(q)}$ in equation (9) yields

$$\zeta(q) = -qa \left\{ \alpha(a) + \frac{(q-1)}{2}\gamma^2(a) \right\}.$$

Imposing $\zeta(q)$ independent on a for all q leads to the specification of α and γ. They are found to be

$$\alpha(a) = \frac{c_1}{a}$$

$$\gamma(a) = \frac{c_2}{\sqrt{a}}$$

where c_1, c_2 are constants. Therefore, we find the well-known form $\zeta(q)$ in the log-normal case

$$\zeta(q) = -q(c_1 - c_2^2/2) - c_2^2 \frac{q^2}{2} = -qm - \frac{\sigma^2 q^2}{2}.$$

Then, using equations (7-8), the propagator can be explicitly written as

$$\Phi_{a,a_0} = \exp\left\{ m \log\left(\frac{a}{a_0}\right) + W\sqrt{\frac{\sigma^2}{2}\log\left(\frac{a}{a_0}\right)} \right\}$$

where W is a zero mean Gaussian random variable with variance 1.

The conclusion is thus the equivalence between the Markovian description of the cascade

$$\mathrm{d}\mathcal{D}_a = \frac{c_1 \mathcal{D}_a}{a}\mathrm{d}a + \frac{c_2 \mathcal{D}_a}{\sqrt{a}}\mathrm{d}w_a$$

and the scale-invariant log-normal approach using Castaing's propagator.

4 Non scale-invariant cascades

In recent works [18,6], Arnéodo *et al.* have exhibited the non scale-invariance of the cascade in fully developed turbulence. This result, obtained on two different flows, is confirmed by the work of Chesnais *et al.* [7]. One can then wonder if, in the non scale-invariant case, the propagator approach and the Markovian description are still equivalent. The aim of the section is to answer the question.

Scale-invariance is revealed by the scaling of the statistics of the propagator. Precisely, the cascade is scale-invariant if the propagator depends only on scales a and a_0 through $\log(a/a_0)$. Therefore, scale-invariance is linked to the usual law of composition of scale $\odot$, viz. $a_1 \odot a_2 = a_1 \times a_2$. Arnéodo *et al.* call the cascade continuously self-similar if the statistics of the propagator depend only on $s(a) - s(a_0)$, where $s(.)$ is a monotonic function. Such a definition is linked to a more general law of composition of scale, viz. $a_1 \odot a_2 = s^{-1}(s(a_1) + s(a_2))$. It can be shown that $(\mathbb{R}^{+*}, \odot)$ is a group.

In this context, the propagator reads

$$\Phi_{a,a_0} = \exp\left\{ m(s(a) - s(a_0)) + \sqrt{\frac{\sigma^2}{2}(s(a) - s(a_0))}W \right\}.$$

Using equations (7, 8) leads to the specification of α and γ which must be written as

$$\alpha(a) = c_1 \dot{s}(a)$$

$$\gamma(a)^2 = c_2^2 \dot{s}(a)$$

where a dot stands for derivation. Therefore, in the non scale-invariant case, the propagator and the Markovian descriptions are still equivalent.

As an example, consider the experimental result obtained by Arnéodo *et al.* [18,6]. For two different flows, function $s(a) = (1 - a^{-\alpha})/\alpha$ provides a close explanation of the propagator (the graph of $m(a, a')$ *versus* $s(a) - s(a')$ is linear, see equation (7). Furthermore, the higher the Reynolds number, the lower the exponent. This case is equivalent to the stochastic differential equation

$$\mathrm{d}\mathcal{D}_a = \frac{c_1 \mathcal{D}_a}{a^{\alpha+1}} \mathrm{d}a + \frac{c_2 \mathcal{D}_a}{\sqrt{a^{\alpha+1}}} \mathrm{d}w_a$$

which of course gives the scale-invariant cascade if $\alpha = 0$.

5 Discussion

In this short paper, we have established the correspondence between Castaing's description of the cascade in turbulence, and Friedrich and Peinke's ideas on the Markovian cascade.

First, note that the correspondence was predictive. Indeed, the multiplicative process defined by Bernard Castaing is a Markov process! However, the description of Friedrich and Peinke is interesting since it provides a continuous evolution in scale. Furthermore, we have shown that results obtained by Arnéodo *et al.*, the non scale-invariant cascade, can be naturally included in the Markovian description. It suffices to play with the drift and diffusion coefficients.

We come back to the problem of time. It has been completely omitted in the development. Hence, the framework discussed here is only descriptive (note however that transposing the ideas of the propagator to the discrete wavelet transform leads to constructive algorithms [19,18,20]). An interesting extension would be to integrate time in the stochastic differential equation, but this problem is tedious. For example, the drift and diffusion coefficients could depend on time, but we would have to check that the solution of the stochastic differential equation is a wavelet transform.

The approach discussed here is however limited to the log-normal case. Other statistics (*e.g.* log-Poisson) cannot be taken into account in this formalism. Note however that point-process-driven stochastic differential equations exist and may be useful in the context. Furthermore, going back to equation (3) and its solution, we see that we have greatly simplified. If the drift and diffusion coefficient are chosen as affine functions instead of purely linear, the solution is no longer log-normal. Looking at (5), affine coefficients lead to a solution that is the superposition of the classic propagation and a correction term which should be worth study.

More precisely, Friedrich and Peinke have shown experimentally that the drift coefficient is linear [8,9]. Therefore, setting $\beta(a) = 0$ in (3) seems reasonable. However, the measurement of the diffusion coefficient reported in [8,9] shows that $\delta(a)$ should not be set to zero. In that case, the Markov description and the propagator approach are no longer equivalent. Furthermore, the higher-order moments of the wavelet coefficients do not scale anymore as $a^{\zeta(q)}$. However, Arnéodo *et al.* have shown in [18,6] that the propagator approach is a very good model of the cascade in the log-normal non scale invariant case. Therefore, it seems that there is a contradiction if $\delta(a) \neq 0$. This leads to the question: Is the experimental determination of $\delta(a)$ in [8,9] sufficiently accurate? Indeed, it is possible that noise induced the non zero $\delta(a)$, since this quantity comes from the estimation of a second order quantity – the second Kramers-Moyal coefficient. We think that the determination of this quantity is important to validate the approaches discussed here, and that much processing of turbulent signals must be carried out.

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